TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 210, 1975

A CLASSIFICATION THEOREM FOR ABELIAN p-GROUPS

BY

R. B. WARFIELD, JR.(1)

ABSTRACT. A new class of Abelian p-groups, called S-groups, is studied, and the groups in this class are classified in terms of cardinal invariants. The class of S-groups includes Nunke's totally projective p-groups. The invariants consist of the Ulm invariants (which Hill has shown can be used to classify the totally projective groups) together with a new sequence of invariants indexed by limit ordinals which are not cofinal with ω. The paper includes a fairly complete discussion of dense isotype subgroups of totally projective p-groups, including necessary and sufficient conditions for two of them to be congruent under the action of an automorphism of the group. It also includes an extension of Ulm's theorem to a class of mixed modules over a discrete valuation ring.

The work reported here was partly announced in [21]. To study the S-groups, it will be necessary to prove a classification theorem for certain modules over the ring Z_p of integers localized at p. We will therefore regard p-groups as torsion modules over this ring, and all modules in this paper will be modules over the ring Z_p. The ring of p-adic integers could have been used just as well. The results are valid for modules over any discrete valuation ring. If A is any Z_p-module, we define $p^\alpha A$ for any ordinal α by $p^{\alpha+1}A = p(p^\alpha A)$, and, if α is a limit ordinal, $p^\alpha A = \bigcap_{\beta<\alpha} p^\beta A$. A module is *reduced* if, for some ordinal α, $p^\alpha A = 0$. The smallest ordinal α such that $p^\alpha A = 0$ is called the *length* of A. $A[p]$ is the submodule of A generated by the elements of order p. We define

$$U_\alpha(A) = (p^\alpha A)[p]/(p^{\alpha+1}A)[p] \qquad (\alpha \geqslant 0)$$

and let $f(\alpha, A) = \dim(U_\alpha(A))$, where the dimension is taken over the field Z/pZ. The cardinals $f(\alpha, A)$ are the *Ulm invariants* of A.

If A is a Z_p-module, a submodule H is *isotype* in A if, for all ordinals α, $p^\alpha H = H \cap p^\alpha A$. If λ is an ordinal, H is λ-dense in A if, for all $\alpha < \lambda$, $A = H + p^\alpha A$. If λ is a limit ordinal, λ is cofinal with ω if there is a sequence λ_i,

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Received by the editors September 24, 1973.

AMS (MOS) subject classifications (1970). Primary 20K10; Secondary 20K30, 20K40.

Key words and phrases. Ulm's theorem, totally projective groups, isotype subgroups, invariants, cotorsion groups, automorphisms of Abelian p-groups.

(1) Research supported in part by N. S. F. grant GP-28946.

$0 < i < \omega$, of smaller ordinals, such that λ is the supremum of the λ_i. We can now state a theorem which summarizes our results on λ-dense, isotype subgroups of totally projective groups.

THEOREM A. *Let G be a totally projective p-group, λ a limit ordinal such that $p^\lambda G = 0$, and H and K λ-dense, isotype subgroups of G.*

(i) *If* $\dim((G/H)[p]) = \dim((G/K)[p])$ *then there is an automorphism of G which takes H isomorphically onto K.*

(ii) *If λ is cofinal with ω, H and K are isomorphic to G.*

(iii) *If λ is not cofinal with ω, then H and K are isomorphic if and only if* $\dim((G/H)[p]) = \dim((G/K)[p])$.

The proof of Theorem A is contained in 1.6 and 3.1 below. It is a consequence of (iii) above that the groups arising in this way are not necessarily totally projective. They are all contained in a larger class, which we call S-groups. To study the S-groups, we first must study a class of mixed Z_p-modules. We give a rather unsatisfactory treatment of these modules, which has the virtue of being brief and of containing all that is needed for the application to p-groups. For a more intrinsic treatment, we refer to [22] and a subsequent paper.

DEFINITION [22]. If λ is a limit ordinal, we say a module M is a λ-elementary KT-module if $p^\lambda M \cong Z_p$ and $M/p^\lambda M$ is torsion and totally projective. A module is a KT-module if it is a direct sum of a totally projective p-group and λ-elementary KT-modules for various λ.

To state the form of Ulm's theorem that holds for KT-modules, we need a new invariant, introduced in [22]. If M is a module, we let $h(0, M) = \dim(M/(pM + T))$ where T is the torsion submodule of M. Note that if M is a direct sum of cyclic modules, then $h(0, M)$ is the number of infinite cyclic summands in a decomposition of M. More generally, $h(0, M)$ is the number of infinite cyclic summands in a basic submodule of M [11, Lemma 21]. For any limit ordinal λ, we define $h(\lambda, M) = h(0, p^\lambda M)$. If M is a KT-module, it is clear that $h(\lambda, M)$ is the number of λ-elementary summands in a decomposition of M.

THEOREM K. *The class $\mathcal{K}$ of KT-modules has the following properties*:

(i) *A module isomorphic to a module in $\mathcal{K}$ is in $\mathcal{K}$.*

(ii) *A direct sum of modules in $\mathcal{K}$ is in $\mathcal{K}$.*

(iii) *If M is a module* (*over the ring Z_p*) *and α an ordinal, then M is in $\mathcal{K}$ if erties* (i), (ii), *and* (iii), *then $\mathcal{C}$ contains $\mathcal{K}$.*

(iv) *If $\mathcal{C}$ is any class of modules containing Z_p and having the closure properties* (i), (ii), *and* (iii), *then $\mathcal{C}$ contains $\mathcal{K}$.*

(v) *If A and B are in $\mathcal{K}$ and $f(\alpha, A) = f(\alpha, B)$ for all ordinals α, and*

$h(\lambda, A) = h(\lambda, B)$ for all limit ordinals λ, then $A \cong B$.

This theorem is proved in 1.3, 4.6, and 5.1 below. The KT-modules possess a number of other nice properties [22], which will be proved in a subsequent paper.

DEFINITION. A p-group is an S-group if it is isomorphic to the torsion submodule of a KT-module. It is a λ-elementary S-group if it is the torsion submodule of a λ-elementary KT-module.

In §2 we will define a family of functors K_λ, where λ is any ordinal. K_λ will associate with any module a Z/pZ-vector space. If λ is a limit ordinal which is not cofinal with ω, we let $k(\lambda, M) = \dim(K_\lambda(M))$. If G is an S-group, we will show that G is a direct sum of a totally projective group and γ-elementary S-groups for various limit ordinals γ which are not cofinal with ω, and that $k(\lambda, G)$ is the number of λ-elementary summands in such a representation.

The invariants $k(\lambda, G)$ and the Ulm invariants will classify the S-groups. To obtain a result for S-groups analogous to Theorem K, we will need a stronger closure property to get an analogue of condition (iv) of that theorem. We say a submodule H of a Z_p-module M is *λ-high* if $H \cap p^\lambda M = 0$, and H is maximal with respect to this property.

THEOREM S. *The class $\mathcal{S}$ of S-groups has the following properties*:

(i) *A group isomorphic to a group in $\mathcal{S}$ is in $\mathcal{S}$.*

(ii) *A direct sum of groups in $\mathcal{S}$ is in $\mathcal{S}$.*

(iii) *If G is a p-group and α an ordinal, then G is in $\mathcal{S}$ if and only if $p^\alpha G$ and $G/p^\alpha G$ are in $\mathcal{S}$.*

(iv) *If G is in $\mathcal{S}$ and λ is a limit ordinal, and H is a λ-high subgroup of G, then H is in $\mathcal{S}$.*

(v) *If $\mathcal{C}$ is any class of p-groups containing Z/pZ and satisfying properties* (i), (ii), (iii), *and* (iv) *above, then $\mathcal{C}$ contains $\mathcal{S}$.*

(vi) *If A and B are in $\mathcal{S}$ and $f(\alpha, A) = f(\alpha, B)$ for every ordinal α and $k(\lambda, A) = k(\lambda, B)$ for every limit ordinal λ which is not cofinal with ω, then $A \cong B$.*

(vii) *If G is in $\mathcal{S}$ then G is totally projective if and only if for every limit ordinal λ which is not cofinal with ω, $k(\lambda, G) = 0$.*

This theorem is an immediate consequence of the definition, and Theorems 3.2, 5.2, and 5.3 below.

A historical remark is in order concerning the form of Theorems K and S. Nunke introduced totally projective p-groups in [16] in order to give a homological description of direct sums of countable p-groups. It was E. Walker who suggested that Ulm's theorem might generalize to this class of groups. Walker showed that if this could be done, then the class of totally projective p-groups would be the smallest class containing Z/pZ and satisfying conditions (i), (ii) and (iii) of Theorem S. Part (iv) of Theorem K and part (v) of Theorem S were motivated by this example. In [19], Parker and Walker proved Ulm's theorem for totally projective p-groups of length less than $\Omega\omega$, thus giving the first major extension of the theory of p-groups to groups of uncountable length. The complete proof of Ulm's theorem for totally projective p-groups required a different technique and an entirely new characterization of these groups, and this was proved by P. Hill (unpublished). A short proof (using Hill's basic ideas) has recently been given by Walker [21].

The theory presented in this paper is still very incomplete. In fact, the results open a veritable Pandora's box of unsolved and, apparently, difficult problems. §6 of this paper contains a list of ten of these.

§1 below uses Hill's methods to classify KT-modules and to construct certain automorphisms for totally projective p-groups. The second section reviews the theory of cotorsion envelopes and defines the functors K_λ. In §3 we classify S-groups and dense isotype subgroups of totally projective groups. In §4, we determine what values the invariants of a KT-module or S-group can have. This includes a new proof of the corresponding result for totally projective p-groups, and a proof that any subgroup of a totally projective p-group has the Ulm invariants of some totally projective p-group. §5 completes the proof of Theorems K and S.

1. Extending isomorphisms and constructing automorphisms.

If G is an Abelian group, H a subgroup and $\phi: G \longrightarrow G/H$ the natural map, we say H is a *p-nice* subgroup if, for all ordinals α, $p^\alpha(G/H) = \phi(p^\alpha G)$.

We remark that if G is a group, $p^\alpha G$ is a p-nice subgroup; and if, for all $i \in I$, H_i is a p-nice subgroup of G_i, then $\bigoplus_{i\in I} H_i$ is p-nice in $\bigoplus_{i\in I} G_i$. These are the only cases we will need to prove our theorems.

We follow E. A. Walker [21] in using Hill's analysis of totally projective p-groups as the definition of these groups.

DEFINITION. If G is a reduced p-group, G is totally projective if it has a family $\mathcal{C}$ of subgroups such that

(i) If $H_i \in \mathcal{C}$, $i \in I$, then $\Sigma_{i\in I} H_i \in \mathcal{C}$,

(ii) If $H \in \mathcal{C}$, and X is a countable subset of G, there is a subgroup $H' \in \mathcal{C}$ such that $H \subseteq H'$, $X \subseteq H'$ and H'/H is countable.

(iii) $\{0\} \in \mathcal{C}$.

(iv) The elements of $\mathcal{C}$ are p-nice subgroups of G.

We now summarize the known results in several theorems.

1.1. THEOREM ([7], [19], [21]). *Let $\mathcal{T}$ be the class of totally projective p-groups.*

(i) *A group isomorphic to a group in $\mathcal{T}$ is in $\mathcal{T}$.*

(ii) *A direct sum of groups in $\mathcal{T}$ is in $\mathcal{T}$.*

(iii) *If G is a p-group, and α an ordinal, then $G \in \mathcal{T}$ if and only if $p^\alpha G \in \mathcal{T}$ and $G/p^\alpha G \in \mathcal{T}$.*

(iv) *If $\mathcal{C}$ is any class of p-groups containing Z/pZ and having properties* (i), (ii) *and* (iii), *then* $\mathcal{C} \supseteq \mathcal{T}$.

(v) *If A and B are in $\mathcal{T}$, and $f(\alpha, A) = f(\alpha, B)$ for all ordinals α, then $A \cong B$.*

Conditions (i) and (ii) are clear. For a proof of (v) we refer to [7] or [21]. The fact that if G is totally projective then $p^\alpha G$ and $G/p^\alpha G$ are also is clear. For the other half of (iii) we remark that the basic extension theorem (1.2 below, proved in [7] and [21]) shows that if $p^\alpha G$ and $G/p^\alpha G$ are totally projective and if the Ulm invariants of G are those of a totally projective group, then G is totally projective. This is easy to see if α is a finite ordinal, and the rest of the proof can be given as in that of 5.1 below. An alternative proof is in [3, 81.9 and 81(A)]. (iv) is an easy consequence of (v) and a suitable existence theory, and a proof of it is contained in our proof of 4.6 below. There are a number of other easily established properties of totally projective groups which we do not list because their generalizations for KT-modules and S-groups are either difficult or unknown. Condition (v) has a generalization which we will need, also essentially due to Hill, and stated in this form in [21].

DEFINITION. If G is a group whose torsion subgroup is a p-group, A is a subgroup, and $A_\alpha = A \cap p^\alpha G$, we define

$$I_\alpha(A) = (A_\alpha + p^{\alpha+1}G)[p]/(p^{\alpha+1}G)[p].$$

We regard $I_\alpha(A)$ as a subspace of the Ulm invariant $U_\alpha(G)$.

1.2. THEOREM ([7], [21]). *If G and H are groups, A and B p-nice subgroups of G and H respectively such that G/A and H/B are totally projective p-groups, $\phi\colon A \longrightarrow B$ an isomorphism such that, for all α, $\phi(A_\alpha) = B_\alpha$, and if, for all α, $U_\alpha(G)/I_\alpha(A) \cong U_\alpha(H)/I_\alpha(B)$, then ϕ extends to an isomorphism of G onto H.*

1.3. COROLLARY. *If M and N are KT-modules, $f(\alpha, M) = f(\alpha, N)$ for all ordinals α, and $h(\lambda, M) = h(\lambda, N)$ for all limit ordinals λ, then $M \cong N$.*

PROOF. We write $M = T \oplus (\bigoplus_{i\in I} M_i)$, where T is a totally projective p-

group and M_i is a $\lambda(i)$-elementary KT-module. For each $i \in I$, choose x_i to be a generator of $p^{\lambda(i)}M_i$ and let $X = \{x_i: i \in I\}$. We similarly choose a subset $Y = \{y_j: j \in J\}$ of N. For a given ordinal λ, the number of elements of I with $\lambda(i) = \lambda$ is exactly $h(\lambda, M)$. There is, therefore, a bijective map $\phi: I \longrightarrow J$ such that $\lambda(i) = \lambda(\phi(i))$ for all $i \in I$.

From the comments following the definition of p-nice subgroups, it is clear that the submodules $[X]$ and $[Y]$, generated by X and Y, are p-nice submodules of M and N, and it is clear by the construction that $M/[X]$ and $N/[Y]$ are totally projective p-groups. It is easily computed that for all α, $I_\alpha([X]) = 0$ and $I_\alpha([Y]) = 0$. Since X and Y are free generators of the free modules $[X]$ and $[Y]$, ϕ defines an isomorphism ϕ': $[X] \longrightarrow [Y]$. It is now clear that all the hypotheses of 1.2 are satisfied, from which we conclude that $M \cong N$.

1.4. LEMMA. *If* $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *is an exact sequence of modules, then multiplication by p in B induces a homomorphism* σ: $C[p] \longrightarrow A/pA$. *If we regard A as a submodule of B and* ν: $B \longrightarrow C$ *is the homomorphism of the sequence, then* $\ker(\sigma) = \nu(B[p])$. *For any ordinal* $\alpha > 0$, *if* $A \subseteq p^\alpha B$ *then* σ *is surjective and* $C[p] = \ker(\sigma) + (p^\beta C)[p]$ *for all* $\beta < \alpha$. *If H is a submodule of B maximal with respect to the property that* $H \cap A = 0$, *then* $\nu(H)[p] = \ker(\sigma)$.

PROOF. We define σ as follows: if $x \in C[p]$ choose a $y \in B$ such that $\nu(y) = x$ and let $\sigma(x) = py + pA$. This is clearly well defined, and it is clear that $\ker(\sigma) = \nu(B[p])$. If $\alpha > 0$, then $A \subseteq p^\alpha B$ if and only if for $\beta < \alpha$ and for any $x \in A$, there is a $y \in p^\beta B$ such that $py = x$. This implies that $x + pA = \sigma(\nu(y))$ and $\nu(y) \in \nu(p^\beta B)[p] \subseteq (p^\beta C)[p]$, which proves all of the result but the last statement, which is trivial.

If M is a Z_p-module and N a submodule, we follow Nunke in saying that N is p^λ*-pure* in M if the corresponding extension $0 \longrightarrow N \longrightarrow M \longrightarrow C \longrightarrow 0$ is in $p^\lambda \operatorname{Ext}(C, N)$.

1.5. LEMMA. *Let G be a p-group and H a subgroup,* λ *a limit ordinal,* $\lambda > 0$, *and suppose that* $p^\lambda G = 0$. *Then the following conditions are equivalent.*

(i) *H is* λ*-dense and isotype.*

(ii) *H is* p^λ*-pure and G/H is divisible.*

(iii) $H \cap pG = pH$ *and, for all ordinals* $\alpha < \lambda$, $G[p] = H[p] + (p^\alpha G)[p]$.

(iv) *There is a* Z_p*-module M and an imbedding* ϕ: $H \longrightarrow M$ *and a surjective homomorphism* ψ: $M \longrightarrow G$ *such that* $\ker(\psi) = p^\lambda M$, $\phi(H)$ *is* λ*-high in M, and* $\psi\phi$ *is the natural imbedding of H into G.*

(v) *There is a* Z_p*-module M and an imbedding* ϕ: $H \longrightarrow M$ *and a surjective homomorphism* ψ: $M \longrightarrow G$ *such that* $\ker(\psi) = p^\lambda M$, $p^\lambda M$ *is free,* $\phi(H)$ *is the* *torsion submodule of M, and $\psi\phi$ is the natural imbedding of H into G.*

PROOF. The equivalence of (i) and (iii) seems to be due to R. S. Pierce [20]. We show that (i) implies (iii). The first condition of (iii) is trivial. For the second, by λ-density, if $x \in G[p]$, there is a $y \in H$ and a $z \in p^\alpha G$ such that $x = y - z$. It follows that $py = pz \in p^{\alpha+1}G$, so, since H is an isotype subgroup, $py \in p^{\alpha+1}H$, so there is a $y' \in p^\alpha H$ such that $py = py'$. Hence, $x = (y + y') - (z + y')$, where $y + y' \in H[p]$ and $z + y' \in (p^\alpha G)[p]$.

The equivalence of (ii) and (iii) is due to Nunke. (ii) implies (iii) by [16, 2.9]. Conversely, if (iii) holds, then since $pH = H \cap pG$, any element in $(G/H)[p]$ is in the image of $G[p]$, and hence in the image of $(p^\alpha G)[p]$ for any $\alpha < \lambda$. It follows that G/H is divisible, from which (ii) follows, again by [16, 2.9].

That (iii) implies (v) seems to be due (in an equivalent form) to Kulikov [12]. For a proof see also [8, 1.7]. (The basic idea is to look at the sequence $0 \longrightarrow H \longrightarrow G \longrightarrow D \longrightarrow 0$ and an epimorphism $f\colon L \longrightarrow D$, where L is divisible and torsion-free and $\mathrm{Ker}(f)$ is free. The induced extension $0 \longrightarrow H \longrightarrow M \longrightarrow L \longrightarrow 0$ works.) (v) implies (iv) trivially.

To complete the proof, we will show that (iv) implies (i). By [10, Theorem 2], $\phi(H)$ is an isotype submodule of M. We recall that if G is a group and N a subgroup, H is N-high [9] in G if $H \cap N = 0$ and H is maximal with respect to this property. It is easy to verify that this implies that the image of N is essential in G/H, and if $N \subseteq p^\omega G$, it follows from this that G/N is p-divisible. It is clear that if H is λ-high in M, then for any $\alpha < \lambda$, $H \cap p^\alpha M$ is $(p^\lambda M)$-high in $p^\alpha M$. Since λ is a limit ordinal, the elements of $p^\lambda M$ still have infinite height in $p^\alpha M$ for any $\alpha < \lambda$, which implies that $p^\alpha M/(\phi(H) \cap p^\alpha M)$ is divisible. Therefore, the image of $p^\alpha M$ in $M/\phi(H)$ is all of $M/\phi(H)$ (since it is a divisible submodule of $M/\phi(H)$ containing the image of $p^\lambda M$, an essential submodule). This implies that $M = \phi(H) + p^\alpha M$, which implies that $G = H + p^\alpha G$, since the map ψ is surjective.

1.6. THEOREM. *Let G be a totally projective p-group of length λ, λ a limit ordinal. Let H and K be two λ-dense, isotype subgroups of G. Then there is an automorphism σ of G such that $\sigma(H) = K$, if and only if $H[p]$ and $K[p]$ have the same codimension in $G[p]$.*

PROOF. The condition on H and K is clearly necessary.

By 1.5, there are modules M and N and imbeddings $\phi\colon H \longrightarrow M$, $\phi'\colon K \longrightarrow N$, such that $p^\lambda M$ and $p^\lambda N$ are free, $\phi(H)$ and $\phi'(K)$ are the torsion submodules of M and N, and there are homomorphisms $\psi\colon M \longrightarrow G$ and $\psi'\colon N \longrightarrow G$, satisfying the conditions of 1.5(v). If $F = p^\lambda M$, then multiplication by p induces an isomorphism $G[p]/H[p] \longrightarrow F/pF$, by 1.4, so that if $H[p]$ and $K[p]$ have the same codimension, then $p^\lambda M \cong p^\lambda N$, since they are free and of the same rank. The conditions of 1.2 are satisfied, so we get an isomorphism of M onto N, which clearly carries $\phi(H)$ onto $\phi'(K)$. The isomorphism between M and N induces an isomorphism $M/p^\lambda M \longrightarrow N/p^\lambda N$, and since ψ and ψ' can be used to identify both of these modules with G, we have an induced automorphism of G with the desired properties.

Theorem 1.6 admits of a generalization, using the λ-Zippin property.

DEFINITION. If M is a module, λ a limit ordinal, and $p^\lambda M = 0$, then M has the λ-Zippin property if for every triple (G, H, f), where G and H are modules, $G/p^\lambda G \cong H/p^\lambda H \cong M$ and $f\colon p^\lambda G \longrightarrow p^\lambda H$ is an isomorphism; there is an extension of f to an isomorphism $f'\colon G \longrightarrow H$.

1.7. LEMMA. *If G is an S-group or a KT-module, and $p^\lambda G = 0$, then G has the λ-Zippin property.*

This is proved by Nunke in [18]. The proof of 1.6 immediately implies the following.

1.8. THEOREM. *If G is a p-group, λ a limit ordinal, $p^\lambda G = 0$, G has the λ-Zippin property, H and K are λ-dense, isotype subgroups of G, and* $\dim((G/H)[p]) = \dim((G/K)[p])$, *then there is an automorphism of G taking H onto K.*

2. Cotorsion envelopes and the functors K_λ. In this section we will define a family of functors K_λ, where λ is any ordinal. We will need the theory of cotorsion modules, which were introduced independently by Fuchs [2], Nunke [14] and Harrison [4]. We follow Fuchs in saying that a module C is cotorsion if and only if $\mathrm{Ext}(F, C) = 0$ for all torsion-free modules F. (It is sometimes required that a cotorsion module be reduced.) If we identify a module M with $\mathrm{Hom}(Z_p, M)$, then we get a natural map $M \longrightarrow \mathrm{Ext}(Q/Z_p, M)$, which is injective if M is reduced. If M is reduced, we let $c(M) = \mathrm{Ext}(Q/Z_p, M)$, and regard M as contained in $c(M)$. We refer to Nunke [14] or Harrison [4] for the proofs that $c(M)$ is cotorsion and reduced and that $c(M)/M$ is torsion-free and divisible. This last implies easily that M is isotype in $c(M)$, and that the torsion submodule of $c(M)$ is contained in M. We recall that a submodule N of M is *pure* in M if $p^n N = N \cap p^n M$ for all n, $0 \leqslant n < \omega$.

2.1. LEMMA. *If M is a reduced module and N a submodule such that M/N is reduced, then the sequence*

$$0 \longrightarrow c(N) \longrightarrow c(M) \longrightarrow c(M/N) \longrightarrow 0$$

is exact. If M is cotorsion, then so is N. In particular, if M is a reduced cotorsion module then so is $p^\lambda M$, for any ordinal λ.

PROOF. The first follows from the exact sequence for Ext and the fact that $\mathrm{Hom}(Q/Z_p, M/N) = 0$. For the second statement, we use the fact that a module C is cotorsion if and only if $\mathrm{Ext}(Q, C) = 0$ (since this implies that $\mathrm{Ext}(D, C) = 0$ for any torsion-free divisible module D, and any torsion-free module can be imbedded in such a D). If $\mathrm{Ext}(Q, M) = 0$, then since $\mathrm{Hom}(Q, M/N) = 0$, $\mathrm{Ext}(Q, N) = 0$.

2.2. LEMMA. *Let M be a reduced module and N a submodule such that M/N is torsion-free and divisible. Then* (i) *if C is a reduced cotorsion module and $f: N \longrightarrow C$ a homomorphism, there is a unique extension of f to a homomorphism $g: M \longrightarrow C$,* (ii) *if f is injective and $C/f(N)$ is torsion-free, then g will also be injective, and* (iii) *if M is cotorsion then there is a natural isomorphism $M \cong c(N)$.*

PROOF. We have an exact sequence

$$\mathrm{Hom}(M/N, C) \longrightarrow \mathrm{Hom}(M, C) \longrightarrow \mathrm{Hom}(N, C) \longrightarrow \mathrm{Ext}(M/N, C).$$

The hypotheses imply that the extremal terms of this sequence vanish, which implies (i). For (ii), we remark that the kernel K of g will be isomorphic to a subgroup of M/N, and if $C/f(N)$ is torsion-free, then $M/N + K$ will also be torsion-free, so K must be isomorphic to a pure subgroup of M/N. A pure subgroup of M/N is divisible, and since M has no divisible subgroups except 0, $K = 0$. For (iii) we apply (i) in both directions to the imbeddings of N into M and $c(N)$.

A torsion-free module is cotorsion if and only if it is algebraically compact [11], which means that it is a direct sum of a divisible module and a module which is complete and Hausdorff in its p-adic topology. Part (iii) of 2.2 implies that if M is a reduced torsion-free module, then $c(M)$ can be identified with the p-adic completion of M. We recall that if A is a torsion-free, reduced module which is complete in its p-adic topology, and $\nu: A \longrightarrow A/pA$ is the natural map, and if X is a subset of A which is taken by ν bijectively onto a Z/pZ-basis of A/pA, then the submodule $[X]$ generated by X is free, with X as a set of free generators, and A is the p-adic completion of $[X]$. In particular, if M is a module such that $p^\lambda M = 0$, then $p^\lambda c(M)$ is complete, torsion-free module, since it is cotorsion by 2.1 and torsion-free (since $M \cap p^\lambda c(M) = p^\lambda M = 0$). The above analysis then motivates the following definition.

DEFINITION. If M is a Z_p-module, and λ is an ordinal, we define

$$K_\lambda(M) = p^\lambda c(M/p^\lambda M)/p^{\lambda+1}c(M/p^\lambda M).$$

If we let $k(\lambda, M)$ be the Z/pZ dimension of $K_\lambda(M)$, we obtain a new family of invariants indexed by ordinals. We remark that in [5] Harrison defined a new sequence of invariants which, in our terminology, were the invariants $h(\lambda, c(M))$. In general it is easy to verify that $h(\lambda, c(M)) = h(\lambda, M) + k(\lambda, M)$, so our present invariants are more delicate than Harrison's for modules in general, but equivalent for p-groups.

Though we define K_λ for any ordinal λ, it is clear that if $0 < n < \omega$, then K_λ and $K_{\lambda+n}$ are equivalent functors, so that for most purposes we can restrict ourselves to limit ordinals.

2.3. LEMMA. *Let M be a reduced Z_p-module and $\lambda > 0$ a limit ordinal which is not cofinal with ω. Suppose that $M/p^\lambda M$ is a totally projective p-group and $p^\lambda M$ is torsion-free. Then there is a natural isomorphisms $p^\lambda M/p^{\lambda+1}M \longrightarrow K_\lambda(H)$ where H is the torsion submodule of M.*

PROOF. M/H is divisible and torsion-free, so by 2.2, there is an imbedding $\phi: M \longrightarrow c(H)$. Since M/H is torsion-free and divisible, $c(H)/M$ is torsion-free and divisible, so $p^\lambda M$ is an isotype submodule of $p^\lambda c(H)$. Since $p^\lambda c(H)$ is reduced and torsion-free, it is complete in its p-adic topology, and if L is the p-adic completion of the pure submodule $p^\lambda M$, then L is a summand of $p^\lambda c(H)$ by [11, Theorem 23]. The quotient $c(H)/L$ is a reduced cotorsion module with $M/p^\lambda M$ as its torsion submodule. If $C = c(H)/L$ and G is the image in C of $M/p^\lambda M$, then C/G is torsion-free and divisible, so we can identify C with $c(G)$ by 2.2. Since G is totally projective, it is a direct sum of groups of smaller length, so by [16, 3.10], $p^\lambda c(G) = 0$ (using the fact that λ is not cofinal with ω). Hence $L = p^\lambda c(H)$, which gives the desired result.

It is clear that the functors K_λ are additive in the sense that there is a natural isomorphism $K_\lambda(M \oplus N) = K_\lambda(M) \oplus K_\lambda(N)$. This additivity does not extend to infinite direct sums, as trivial examples show. However, if λ is a limit ordinal which is not cofinal with ω, Nunke has shown that K_λ is strongly additive in this sense. We will not need this general result, since the special case which occurs when all of the modules involved are S-groups is quite easy.

3. **Classification theorems.** We classify dense isotype subgroups of totally projective p-groups up to isomorphism, and use this to classify S-groups. The groups classified in 3.1 actually are S-groups, but to prove this, one needs some considerations concerning admissible invariants which we postpone till §4.

3.1. THEOREM. *Let G be a totally projective p-group of length λ, λ a limit* *ordinal. If λ is cofinal with ω then all λ-dense, isotype subgroups of G are isomorphic to G. If λ is not cofinal with ω, then two λ-dense, isotype subgroups H and K are isomorphic if and only if $H[p]$ and $K[p]$ have the same codimension in $G[p]$.*

PROOF. If λ is cofinal with ω, then by Nunke's theorem [16, 4.4], $p^\lambda \mathrm{Ext}$ is hereditary, which implies that any p^λ-pure subgroup of G is p^λ-projective. If H is a λ-dense, isotype subgroup of G, then H is p^λ-projective by 1.5, which certainly implies that H is p^α-projective for any $\alpha > \lambda$. If $\alpha < \lambda$, then $H/p^\alpha H \cong G/p^\alpha G$, so $H/p^\alpha H$ is p^α-projective. This shows that H is totally projective, and since its Ulm invariants are the same as those of G, $H \cong G$.

To show the second half of the theorem, we must show that if H is a λ-dense, isotype subgroup of G and is not cofinal with ω, then the codimension of $H[p]$ in $G[p]$ is an isomorphism invariant of H, and the rest will follow by Theorem 1.6. Using part (v) of 1.5, we construct a Z_p-module M and a mapping ψ: $H \longrightarrow M$ taking H isomorphically onto the torsion submodule of M. We also have a surjective mapping ψ: $M \longrightarrow G$ with kernel $p^\lambda M$ such that $\psi\phi$ is the natural imbedding of H into G. It follows, from 1.4, that there is a natural isomorphism $G[p]/H[p] \longrightarrow p^\lambda M/p^{\lambda+1}M$, given by multiplication by p. Since λ is not cofinal with ω, 2.3 implies that $p^\lambda M/p^{\lambda+1}M \cong K_\lambda(H)$, which shows that the codimension of $H[p]$ in $G[p]$ is indeed an isomorphism invariant of H.

DEFINITION. If M is a module and λ is a limit ordinal which is not cofinal with ω, we define $k(\lambda, M) = \dim(K_\lambda(M))$, where $K_\lambda(M)$ is regarded as a Z/pZ-vector space.

3.2. THEOREM. *If G and H are S-groups, then $G \cong H$ if and only if for every ordinal α and every limit ordinal λ which is not cofinal with ω, $f(\alpha, G) = f(\alpha, H)$ and $k(\lambda, G) = k(\lambda, H)$. Any S-group G is a direct sum of a totally projective p-group and λ-elementary S-groups where the ordinals λ are not cofinal with ω. The number of λ-elementary summands in such a representation is exactly $k(\lambda, G)$. In particular, G is totally projective if and only if $k(\lambda, G) = 0$ for all limit ordinals λ which are not cofinal with ω.*

PROOF. We first remark that if G is totally projective then $k(\lambda, G) = 0$ for all limit ordinals λ which are not cofinal with ω, by [16, 3.10], since $G/p^\lambda G$ is a direct sum of groups of smaller length. We remark that this makes the last statement of the theorem a trivial consequence of the rest.

We next complete the proof of the second half of the theorem. Any S-group is a direct sum of a totally projective group and λ-elementary S-groups for various ordinals λ, by the definition. By 3.1, if λ is a limit ordinal which is cofinal with ω, then a λ-elementary S-group is totally projective. Therefore, any S-group is the direct sum of a totally projective group and λ-elementary S-groups, where the λ's may all be assumed to be not cofinal with ω. If Λ is the set of limit ordinals (not dofinal with ω) for which summands appear, then we can write $G = T \oplus (\bigoplus_{\lambda\in\Lambda} G_\lambda)$ where each G_λ is a direct sum of λ-elementary S-groups. Fixing λ, we write $G = T \oplus A \oplus G_\lambda \oplus B$, where A is the direct sum of the G_γ for $\gamma > \lambda$ and B is the direct sum of the G_γ for $\gamma < \lambda$. $k(\lambda, T) = 0$ by the first remark of the proof. $k(\lambda, B) = 0$, since $p^\lambda c(B) = 0$. $k(\lambda, A) = 0$ since if E is a γ-elementary S-group and $\lambda < \gamma$, then $E/p^\lambda E$ is totally projective. Hence $k(\lambda, G) = k(\lambda, G_\lambda)$. G_λ is the torsion submodule of a KT-module M_λ such that $p^\lambda M_\lambda$ is a free module whose rank is the number of λ-elementary summands of G_λ. By 2.3, $k(\lambda, G)$ is equal to the rank of $p^\lambda M_\lambda$, which establishes the computation of $k(\lambda, G)$ stated in the second part of the theorem.

We turn now to the first part of the theorem. We have shown that an S-group G can be regarded as the maximal torsion subgroup of a KT-module M, where M is the direct sum of a totally projective p-group and λ-elementary KT-module, $\lambda > 0$, where the ordinals λ are not cofinal with ω, and if this is done, then the number of λ-elementary summands of M is just $k(\lambda, G)$. It is also $h(\lambda, M)$. We note also that the Ulm invariants of M and G agree, and that if λ is a limit ordinal which is cofinal with ω, then $h(\lambda, M) = 0$. Hence we can imbed G and H as the torsion submodules of KT-modules M and N with the same invariants. By 1.3, $M \cong N$, and since G and H are the maximal torsion submodules, $G \cong H$.

4. Admissible sets of invariants.

DEFINITION. Let f and h be functions associating to each ordinal number a cardinal number, and such that for some ordinal γ, $f(\alpha) = h(\alpha) = 0$ for all α, $\alpha > \gamma$. Let

$$\lambda(f, h) = \sup\{\alpha + 1, f(\alpha) \neq 0; \alpha + \omega, h(\alpha) \neq 0\}.$$

We say (f, H) is an *admissible pair* if $h(\alpha) = 0$ unless α is a limit ordinal, and if α is a limit ordinal with $\alpha + \omega < \lambda(f, h)$, and $0 \leqslant n < \omega$, then

$$(*) \qquad \sum_{\alpha+n\leqslant\beta<\alpha+\omega} f(\beta) \geqslant \sum_{\alpha+\omega\leqslant\beta<\lambda(f,h)} (f(\beta) + h(\beta)).$$

4.1. THEOREM. *If f and h are functions associating to each ordinal a cardinal number, then there is a KT-module M such that $f(\alpha, M) = f(\alpha)$ and $h(\beta, M) = h(\beta)$ for all ordinals α and all limit ordinals β if and only if (f, h) is an admissible pair.*

Before proving this theorem, we investigate a large class of modules whose Ulm invariants satisfy the above inequalities. The next lemma is motivated by a method used by Nunke in [15, pp. 167–168]. For any module A, we let $\#(A)$ be the smallest cardinal number which is the cardinality of a set of generators of A. (For the countable ring Z_p, this number is just the cardinality of A, except when A is finite, but the version presented here is valid over any discrete valuation ring.)

DEFINITION. A module M is said to be in the class Θ if and only if for every submodule A of M, $\#(A) = \#(A/pA)$.

4.2. LEMMA. *If M_i, $i \in I$, are in the class Θ then $\bigoplus_{i\in I} M_i$ is also in Θ. If M is a module and C a submodule which is finitely generated, and if M/C is in Θ, then M is in Θ.*

PROOF. In the first case, let A be a submodule of $\bigoplus_{i\in I} M_i$. Let X be a subset of A which is taken bijectively onto a basis of A/pA under the natural map, and let $B = [X]$. Clearly, $\#(B) = \#(A/pA)$. Let π_i be the projection of M onto the summand M_i. Since $A = B + pA$, $\pi_i(A) = \pi_i(B) + p\pi_i(A)$, and since $M_i \in \Theta$, $\#(\pi_i(A)) = \#(\pi_i(B))$. If B is not finitely generated, then $\#(B) = \Sigma_{i\in I} \#(\pi_i(B))$, and A is also infinitely generated, so $\#(A) = \Sigma_{i\in I} \#(\pi_i(A))$ and the result follows. If B is finitely generated, then since $\pi_i(A)/\pi_i(B)$ is divisible, and since it is clear that an element of Θ cannot have a divisible submodule, $\pi_i(A) = 0$ for all but finitely many i, since the same holds for B. Since each $\pi_i(A)$ is finitely generated, it follows that A must be finitely generated, in which case a standard computation shows that $\#(A) = \#(A/pA)$.

For the second statement, we may restrict ourselves to infinitely generated submodules of M entirely. If $\phi\colon M \longrightarrow M/C$ is the natural map, and A is a submodule of M which is not finitely generated, then $\#(A) = \#(\phi(A)) = \#(\phi(A)/p\phi(A)) \leqslant \#(A/pA)$, from which the result follows.

4.3. LEMMA. *If $M \in \Theta$ and for any ordinal α and limit ordinal β we define $f(\alpha) = f(\alpha, M)$ and $h(\beta) = h(\beta, M)$, and set $h(\alpha) = 0$ for any ordinal α which is not a limit ordinal, then (f, h) is an admissible pair.*

PROOF. All that we need to prove is the inequality (*) of the definition. If α is a limit ordinal, we can write $p^\alpha M = B \oplus A$ where $p^n B = 0$ and A has no summands which are cyclic of order less than p^{n+1}. Let $\nu\colon A \longrightarrow A/p^{\alpha+\omega}M$ be the natural map and let N be the submodule consisting of all elements of A which are mapped into the torsion submodule of $A/p^{\alpha+\omega}M$. We note that $p^{\alpha+\omega}M = p^\omega A = p^\omega N$, and if $k < \omega$, we compute the Ulm invariants $f(k, N) = 0$, $k < n$, and $f(k, N) = f(\alpha + k)$, $k \geqslant n$. From this, it is clear that

$$\#(N/pN) = \sum_{\alpha+n\leqslant\beta<\alpha+\omega} f(\beta).$$

Since $N \in \Theta$ and $p^{\alpha+\omega}M \subseteq N$, we can prove the inequality by finding a submodule C of $p^{\alpha+\omega}M$ such that

$$\#(C) = \sum_{\alpha+\omega\leqslant\beta<\lambda} (f(\beta) + h(\beta))$$

(since $\#(C) \leqslant \#(N) = \#(N/pN)$). (Here, λ is the length of M.)

If γ is a limit ordinal and B_γ is a basic submodule of $p^\gamma M$ then

$$\#(B_\gamma) = \sum_{\gamma\leqslant\beta<\gamma+\omega} (f(\beta) + h(\beta)).$$

Since $B_\gamma \cap p^{\gamma+\omega}M = 0$, the submodules B_γ of M are independent, and if we let $C = \bigoplus B_\gamma$, where the direct sum ranges over all limit ordinals γ, $\alpha + \omega \leqslant \gamma < \lambda$, then C has the desired properties.

4.4. LEMMA. *If G is a p-group of length α, there is a Z_p-module M such that $p^\alpha M \cong Z_p$ and $M/p^\alpha M \cong G$.*

This is a special case of a well-known existence theorem, first proved by Pierce (unpublished); see [20], [18, Theorem 1.6].

PROOF OF THEOREM 4.1. The existence proof is by induction on $\lambda = \lambda(f, h)$. We distinguish two cases: (i) when there is a limit ordinal γ such that $\lambda = \gamma + \omega$ or $\lambda = \gamma + n$ for some positive integer n, and (ii) when no such γ exists. Case (ii) is easy. By a standard infinite combinatorial argument, we can find functions f_i and h_i ($i \in I$) such that $f = \Sigma_{i\in I} f_i$, $h = \Sigma_{i\in I} h_i$, and for each $i \in I$, (f_i, h_i) is an admissible pair with $\lambda(f_i, h_i) < \gamma$. By induction, there are KT-modules M_i ($i \in I$) with invariants given by the functions (f_i, h_i), and $M = \bigoplus_{i\in I} M_i$ corresponds to the pair (f, h).

In case (i), we use a similar combinatorial argument to write f and h as sums of functions f_i and h_i ($i \in I$), such that each pair (f_i, h_i) is admissible and we impose the additional restriction that for each $i \in I$,

$$\sum_{\gamma\leqslant\alpha<\gamma+\omega} (f_i(\alpha) + h_i(\alpha)) = 1.$$

By induction, there are KT-modules H_i, $i \in I$, such that $p^\gamma H_i = 0$ and such that for $\alpha < \gamma$, $f(\alpha, H_i) = f_i(\alpha)$; and for limit ordinals β, $\beta < \gamma$, $h(\beta, H_i) = h_i(\beta)$. Let C_i be a cyclic module, $C_i \cong Z_p$ if $h_i(\gamma) \neq 0$ and $C_i \cong Z/p^{n+1}Z$ if $f_i(\gamma + n) \neq 0$. By 4.4, there are KT-modules M_i such that $p^\gamma M_i \cong C_i$ and $M_i/p^\gamma M_i \cong H_i$. The module $M = \bigoplus_{i\in I} M_i$ now satisfies our conditions.

All of the modules produced in the existence part of the proof of 4.1 are in

Θ, by 4.2. If M is any KT-module, we can get from this existence proof a KT-module N in Θ such that the invariants of N and $N \oplus M$ agree, from which it follows (from 1.3) that $N \cong N \oplus M$. Since a summand of an element of Θ is in Θ, this shows that any KT-module is in Θ, thus (with another reference to 4.3) completing the proof of 4.1.

4.5. COROLLARY. *Let* $\mathcal{C}$ *be a class of modules containing the KT-modules, and such that the direct sum of two modules in* $\mathcal{C}$ *is in* $\mathcal{C}$. *Suppose that for all pairs of elements M and N of* $\mathcal{C}$, $M \cong N$ *if for all ordinals* α *and all limit ordinals* λ, $f(\alpha, M) = f(\alpha, N)$ *and* $h(\lambda, M) = h(\lambda, N)$. *Then* $\mathcal{C}$ *coincides with the class of KT-modules.*

PROOF. If M is in $\mathcal{C}$ there is a KT-module N such that the invariants of N and $N \oplus M$ are the same. It follows that $N \oplus M \cong N$, so that M is a summand of a KT-module, and hence in Θ. M therefore has the same invariants as some KT-module K, from which it follows by hypothesis that $M \cong K$, thus proving the result.

4.6. COROLLARY. *Let* $\mathcal{C}$ *be a class of* Z_p*-modules such that*

(i) *any module isomorphic to an element of* $\mathcal{C}$ *is in* $\mathcal{C}$;

(ii) *a direct sum of modules in* $\mathcal{C}$ *is in* $\mathcal{C}$;

(iii) *if for some ordinal* α, $p^\alpha G$ *and* $G/p^\alpha G$ *are in* $\mathcal{C}$, *then* $G \in \mathcal{C}$.

Then (a) *if* Z/pZ *is in* $\mathcal{C}$, $\mathcal{C}$ *contains all totally projective p-groups, and* (b) *if* Z_p *is in* $\mathcal{C}$, $\mathcal{C}$ *contains all KT-modules.*

This follows from the proof of 4.1. Notice that in (iii), if we had restricted $p^\alpha G$ to be cyclic of order p, we still would get all totally projective p-groups. Part (a) was proved by Parker and Walker in [19].

4.7. THEOREM. *If G is a p-group which can be imbedded in a totally projective group, then there is a totally projective p-group H such that for all* α, $f(\alpha, G) = f(\alpha, H)$.

PROOF. This is an immediate consequence of 4.1, 4.3, and the fact that all totally projective p-groups are in Θ (which follows from the existence proof in 4.1).

4.8. THEOREM. *Let f and k be two functions associating to each ordinal a cardinal and such that there is some ordinal* τ *such that* $f(\alpha)$ *and* $k(\alpha)$ *vanish for* $\alpha > \tau$. *Suppose also that for any ordinal* α, $k(\alpha) = 0$ *unless* α *is a limit ordinal which is not cofinal with* ω, $\alpha > 0$. *Then there is an S-group G such that* $f(\alpha, G) = f(\alpha)$ *and* $k(\lambda, G) = k(\lambda)$ *for all ordinals* α *and all limit ordinals* λ *which* *are not cofinal with* ω, *if and only if for any limit ordinal and integer* n, $0 \leqslant n < \omega$,

$$\sum_{\gamma+n\leqslant\beta<\gamma+\omega} f(\beta) \geqslant \sum_{\gamma+\omega\leqslant\beta<\tau} (f(\beta) + k(\beta)).$$

PROOF. By the argument used in the proof of 3.2, any S-group is the torsion submodule of a KT-module M such that $f(\alpha, M) = f(\alpha, G)$ for all ordinals α, $h(\lambda, M) = 0$ for any limit ordinal λ which is cofinal with ω, and $h(\lambda, M) = k(\lambda, G)$ for any limit ordinal which is not cofinal with ω. 4.1 therefore implies that the above inequality must hold. Conversely, if the above inequality holds, then there is a KT-module M with these invariants, and its torsion submodule is the desired G.

5. Closure properties of the classes of S-groups and KT-modules.

5.1. THEOREM. *If* M *is a* Z_p*-module and* δ *an ordinal, then* M *is a* KT*-module if and only if* $p^\delta M$ *and* $M/p^\delta M$ *are both* KT*-modules.*

PROOF. If M is a KT-module, it is clear from the definition and the corresponding property for totally projective groups that $p^\delta M$ and $M/p^\delta M$ are KT-modules.

For the converse, we first remark that we may assume that δ is a limit ordinal. This is because we already know the corresponding result for totally projective p-groups, which makes it clear from the definition that if M is a module and $p^n M$ is a KT-module, then so is M.

We next show that the invariants of M are those of some KT-module. Let σ: $(M/p^\delta M)[p] \longrightarrow p^\delta M/p^{\delta+1}M$ be the surjective homomorphism defined in Lemma 1.4. Let L be a subgroup of $(M/p^\delta M)[p]$ which is mapped isomorphically onto $p^\delta M/p^{\delta+1}M$ by σ. Clearly, since $p^\delta M \in \Theta$,

$$\#(L) = \sum_{\delta\leqslant\beta<\lambda} (f(\beta, M) + h(\beta, M))$$

where λ is the length of M. We wish to check the inequality (*) for a limit ordinal α, $\alpha < \delta$ (since for other limit ordinals, the result is clear). For every limit ordinal γ such that $\alpha < \gamma < \delta$, let B_γ be a basic submodule of $p^\gamma M$ and B'_γ the image of B_γ in $M/p^\delta M$. As in the proof of 4.3, if C is the direct sum of the submodules B'_γ, then

$$\#(C) = \sum_{\alpha+\omega\leqslant\beta<\delta} (f(\beta, M) + h(\beta, M)).$$

Since $C \cap L = 0$ by construction (this is why we did not choose the modules B'_γ as arbitrary basic submodules of $p^\gamma(M/p^\delta M)$), as in the proof of 4.3,

$$\#(C) + \#(L) \leqslant \sum_{\alpha+n\leqslant\beta<\alpha+\omega} f(\beta, M).$$

Putting these inequalities together, we obtain condition (*).

From this we know that there is a KT-module N with the same invariants as M. Since $p^\delta N$ and $p^\delta M$ are KT-modules with the same invariants, there is an isomorphism f: $p^\delta M \longrightarrow p^\delta N$. Since KT-modules of length δ have the δ-Zippin property (1.7), f extends to an isomorphism of M onto N, proving the result.

5.2. THEOREM. *If α is an ordinal and G a group, then G is an S-group if and only if $p^\alpha G$ and $G/p^\alpha G$ are S-groups.*

PROOF. If G is an S-group then G is the torsion subgroup of a KT-module M. $p^\alpha G$ is the torsion subgroup of the KT-module $p^\alpha M$, and therefore is an S-group. If G is a λ-elementary S-group, then if $\alpha \geqslant \lambda$, $G/p^\alpha G = G$, and is therefore an S-group, while if $\alpha < \lambda$, $G/p^\alpha G$ is totally projective. Since any S-group, G, is a direct sum of λ-elementary S-groups and a totally projective group, it follows that $G/p^\alpha G$ is an S-group.

We suppose now that G is a p-group, and that $p^\alpha G$ and $G/p^\alpha G$ are S-groups. Write $\alpha = \delta + n$, where δ is a limit ordinal. If ϕ: $p^\alpha G \longrightarrow M$ is an imbedding of $p^\alpha G$ as the torsion subgroup of a KT-module M (in which it may be assumed that $M/p^\alpha G$ is divisible), then there is a module M' and an imbedding ϕ': $p^\delta G \longrightarrow M'$ such that $p^n M' = M$. By 5.1, M' is a KT-module, and by construction, $p^\delta G$ is its torsion submodule. We therefore need only show that G is an S-group given that $p^\delta G$ and $G/p^\delta G$ are S-groups, where δ is a limit ordinal.

By the same argument as used in the proof of 5.1, the invariants of G are those of an S-group H. $p^\delta G$ and $p^\delta H$ are therefore isomorphic S-groups and there is an isomorphism ϕ: $p^\delta G \longrightarrow p^\delta H$. By 1.7, this extends to an isomorphism of G onto H, proving the result.

5.3. THEOREM. *If G is an S-group and λ a limit ordinal, a λ-high subgroup of G is also an S-group. If G is an S-group, λ a limit ordinal and $p^\lambda G = 0$, then a λ-dense, isotype subgroup of G is an S-group.*

PROOF. By 1.5, the first stated result is a consequence of the second. We assume, then, that $p^\lambda G = 0$ and that H is a λ-dense, isotype subgroup. By 1.5, there is a Z_p-module M and maps ϕ: $H \longrightarrow M$, ψ: $M \longrightarrow G$ such that ϕ imbeds H as the torsion submodule of M, $\ker(\psi) = p^\lambda M$, $p^\lambda M$ is free, ψ is surjective, and $\psi\phi$ is the natural imbedding of H into G.

We can imbed G into a KT-module M' as its torsion submodule, by a map f: $G \longrightarrow M'$. Since the map $\mathrm{Ext}(M', p^\lambda M) \longrightarrow \mathrm{Ext}(G, p^\lambda M)$ is surjective, M is a submodule of a module M'' such that the map ψ: $M \longrightarrow G$ extends to a surjective map $\psi': M'' \longrightarrow M'$ with kernel $p^\lambda M$. Clearly, $p^\lambda M''$ is free and $M''/p^\lambda M'' \cong M'/p^\lambda M'$, so, by 5.1, M'' is a KT-module. By construction, it is clear that ϕ imbeds H as the torsion submodule of M'', so (by definition) H is an S-group.

6. Unsolved problems.

Problem 1. Is a summand of an S-group necessarily an S-group?

Problem 2. If not, can summands of S-groups be classified by the same invariants?

Problem 3. Is there a class C of reduced p-groups such that the direct sum of two groups in C is in C, such that C properly contains the class of S-groups, and such that the groups in C can be classified by the same invariants?

Before Ulm's theorem was proved for totally projective groups, Walker showed [19] that if this could be done, Ulm's invariants could not be used to classify the elements of any larger class of reduced p-groups closed with respect to finite direct sums. His argument shows that any group in such a class C (as described in Problem 3) would be a summand of an S-group. An affirmative answer to Problem 1 would therefore imply a negative answer to Problem 3, but possibly Problem 3 could be answered independently (compare 4.5 above). In [23], Wick gives a "projective" characterization of summands of S-groups.

Problem 4. Is there a reasonable structure theory for a larger class of groups which are isotype subgroups of totally projective groups?

Hill showed [6] that an isotype subgroup of a totally projective group of countable length is again totally projective. However, C. Megibben has pointed out to the author that an isotype subgroup of a direct sum of countable p-groups need not be an S-group. (This is easily obtained from an example in the last paragraph of [13, p. 109], using the fact that if G is an S-group, then $L_\Omega G/G$ is divisible.)

Problem 5. Is there a reasonable structure theory for the groups which arise as $\operatorname{Tor}(A, B)$, when A and B are totally projective?

Nunke showed [14] that if A and B are totally projective of countable length, then $\operatorname{Tor}(A, B)$ is totally projective, but Wick [23] shows that there are totally projective groups A and B such that $\operatorname{Tor}(A, B)$ is not an S-group.

Problem 6. If G is totally projective and H a subgroup such that $(G/H)[p]$ is finite, is H necessarily an S-group?

Edington [1] gives necessary and sufficient conditions, in this situation, for H to be totally projective.

If λ is a limit ordinal and G is a p-group with $p^\lambda G = 0$, we say G has the *λ-Zippin property* if for all pairs H and K of p-groups such that $H/p^\lambda H \cong K/p^\lambda K \cong G$ and all isomorphisms $f\colon p^\lambda H \longrightarrow p^\lambda K$, there is an isomorphism ϕ:

$H \longrightarrow K$ whose restriction to $p^\lambda H$ is f. Nunke shows [18] that if G is an S-group and $p^\lambda G = 0$, then G has the λ-Zippin property. The converse fails in a nontrivial way.

Problem 7. If G is a p-group and for all limit ordinals λ, $G/p^\lambda G$ has the λ-Zippin property, is G necessarily an S-group?

If u is a function associating to each nonnegative integer an ordinal or ∞, and G is a p-group, we define $uG = \{x \in G\colon \text{for all } n \geqslant 0, p^n x \in p^{u(n)}G\}$. Kaplansky showed in [11] that if G is a reduced countable p-group, then the fully invariant subgroups of G are exactly the subgroups uG obtained in this way. The methods now available make it trivial to extend this result to totally projective p-groups. L. Fuchs and E. A. Walker (unpublished) have shown that if G is totally projective and u is such a function, then uG and G/uG are again totally projective.

Problem 8. If G is an S-group, are all fully invariant subgroups of the form uG?

Problem 9. If G is an S-group, and u is one of the above functions, are uG and G/uG necessarily S-groups?

Problem 10. Let G and H be reduced p-groups and $\phi\colon G \longrightarrow H$ a homomorphism such that the induced maps $U_\alpha(G) \longrightarrow U_\alpha(H)$ and $K_\lambda(G) \longrightarrow K_\lambda(H)$ are isomorphisms for all ordinals α and all limit ordinals λ. Is ϕ necessarily an isomorphism?

The answer to Problem 10 is yes for groups with no elements of infinite height.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195